# Capital Allocation and Risk Sharing*

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ABSTRACT

We develop an optimal contracting model in which limited enforcement of financial contracts generates dispersion in marginal products of capital across firms. We show that the optimal contract can be implemented using state-contingent transfers and a simple collateral constraint that limits the capital input of firms by a fraction of the financial wealth of the firm owner. Compared to models with exogenous collateral constraint and incomplete markets (for example Moll (2014)), we find that the degree of measured misallocation is *increasing* in the persistence of the idiosyncratic productivity shocks. Under the optimal contract, the possibility to transfer wealth from high productivity states to low productivity states allows firm owners to trade off efficient allocation of consumption against the efficient allocation of capital. We show that for reasonable values of risk aversion, insurance needs more than offset production efficiency concerns.

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# 1 Introduction

A vast empirical literature (see Hsieh and Klenow (2009) and several others) documents dispersion in marginal products of capital across firms for several countries. These patterns are commonly interpreted as evidence of capital misallocation and responsible for cross-country total factor productivity gaps. To account for the empirically observed capital misallocation, several authors have proposed models of financial frictions where entrepreneurs' borrowing is limited by their wealth. However, existing models typically do not provide an explicit micro-foundation for the financial constraints. In addition, in applied work, these models have had limited success in accounting for the large observed dispersion in marginal products. In this paper, we develop an equilibrium model of investment where financial constraints are derived from agency frictions. We demonstrate that the optimal contract in our setting can be implemented using state-contingent transfers and a collateral constraint similar to the one used in the previous literature. Compared to models with exogenous constraints, we show that our optimal contracting framework amplifies the degree of capital misallocation.

Our environment consists of well-diversified intermediaries who offer long-term contracts to entrepreneurs who own productive technologies but not (enough) wealth. Entrepreneurs are risk averse and subject to idiosyncratic productivity shocks. The optimal contract balances insurance against low productivity states as well as funding investment for high productivity states. The agency friction we consider is limited enforcement of financial contracts – entrepreneurs have an option to renege the current contract, abscond with a fraction of the capital stock, and anonymously enter into a new contract with a financial intermediary.

We obtain two main results in this environment. First, we characterize the optimal lending contract and provide an implementation result. We show that the equilibrium allocation with optimal contracts subject to limited enforcement can be implemented using Arrow securities and a collateral constraint. The constraint is linear in the financial wealth of the entrepreneur, and its tightness (or the slope parameter) is independent of idiosyncratic histories. Our implementation thus mimics the exogenous collateral constraints widely used in applied work but allows entrepreneurs to transfer wealth across states.

Second, we demonstrate that under the optimal contract, measured capital misallocation is increasing in the persistence of idiosyncratic productivity shocks under moderate levels of risk aversion. This result is in sharp contrast with that obtained in models with exogenous collateral constraints, where self-financing undoes capital misallocation in the presence of persistent productivity shocks.

In a seminal work, Moll (2014) shows that in an economy where firm owners' borrowing capacity is determined by their financial wealth and they can borrow and save exclusively using a risk-free bond, the degree of misallocation is decreasing in the persistence of idiosyncratic shocks. In the presence of collateral constraints, capital misallocation occurs if owners of high productivity firms do not accumulate enough wealth to finance the efficient level of capital. When productivity shocks are persistent, owners of high productivity firms typically have experienced a long sequence high productivity, and therefore would have saved out of their financial constraint.

Our result implies that the above intuition depends crucially on the assumption of exogenously incomplete market, that is, the risk-free bond is the only financial asset and entrepreneurs cannot allocate wealth across different productivity states. In our setup, the only friction is limited enforcement; markets are otherwise complete. The optimal contract allows firm owners to trade off the allocation of wealth to insure against adverse income states versus the allocation of wealth to maximize productive efficiency. On the one hand, insurance, i.e., consumption efficiency, implies that entrepreneurs need to borrow from states with high productivity and transfer wealth to states with low productivity. On the other hand, production efficiency requires more wealth in high-productivity states to back the financing of a larger amount of capital and less wealth in low productivity states. These two distinct motives pull in opposite directions. For a given level of risk aversion, as shocks become more persistent, the insurance motives are stronger. Entrepreneurs choose to enter productive states with low levels of wealth; sacrificing productive efficiency in order to attain better consumption insurance. This makes misallocation higher.

Our paper contributes to the literature on optimal contracting and capital misallocation. The optimal contract setup with limited enforcement builds on the classical contributions of Kehoe and Levine (1993), Alvarez and Jermann (2000) and Albuquerque and Hopenhayn (2004). Kehoe and Levine (1993) and Alvarez and Jermann (2000) consider risk sharing problems in endowment economies without production decisions. Albuquerque and Hopenhayn (2004) study optimal lending contracts where entrepreneurs are risk-neutral and focus on production decisions in the presence of limited enforcement. Recently, Rampini and Viswanathan (2010, 2013) study the implications of limited contract enforcement for risk management and capital structure also in the context of risk-neutral agents. There is a parallel literature that studies the impact of limited commitment on consumption risk sharing, for example, Krueger and Perry (2006), Krueger and Perry (2004), and Krueger and Uhlig (2006). In contrast, our setup features risk averse agents in a production economy and we emphasize the trade-off between risk sharing and production efficiency. In addition, our implementation result is novel and provides micro-foundation for the widely used wealth-based collateral constraints used in the literature.

Our paper is related to the literature on capital misallocation. For instance, Banerjee and Moll (2010); Buera et al. (2011); Buera and Shin (2011, 2013); Buera et al. (2015); Midrigan and Xu (2014); Moll (2014). All these papers use a risk-free bond and exogenously-specified collateral constraints. Our paper uses an optimal contracting framework to show that the steady-state level of capital misallocation in the aforementioned papers is to a large extent driven by the assumption of risk-free bond.

On the methodological side, our paper is related to the literature of continuous-time dynamic contracting, especially those focus on limited commitment. Using the continuous-time methodology, Grochulski and Zhang (2011) solve an optimal risk sharing problem with limited commitment in an endowment economy. Ai and Li (2015) study the impact of limited commitment on CEO compensation and investment. Bolton et al. (2019) analyze the implications on limited commitment on corporate liquidity and risk management.

The paper is organized as follows. Section 2 describes the environment and the setup of the model. In Section 3, we present our decentralization result and show that any equilibrium with long-term contracts can be implmented using Arrow securities and a linear collateral constraint. In Section 4, we prove our main result on the relationship between persistence of idiosyncratic risk and capital misallocation. Section 6 concludes. The proofs and other details omitted from the main text are relegated to the Appendix.

## 2 Model Setup

**Preferences** Time is continuous. Households are divided into a unit mass of entrepreneurs and a unit mass of workers. All individuals have expected utility with a common constant coefficient of relative risk aversion, that is, given a consumption process $\{C_t\}_{t=0}^{\infty}$, their preferences are ordered by

$$E_0\left[\int_0^{\infty} e^{-\beta t}\beta\frac{C_t^{1-\gamma}}{1-\gamma}dt\right]. \tag{1}$$

**Endowments and Technology** Entrepreneurs are endowed with an idiosyncratic productivity process $\theta$ and have access to a technology that combines capital and labor to produce output using

$$y = (\theta K)^{\alpha} L^{1-\alpha}, \tag{2}$$

where the process productivity $\theta$ follows

$$d\theta_t = \mu(\theta_t)dt + \sigma(\theta_t)dM_t,$$

with $M_t$ being a Brownian motion.[1] We assume that $\mu(\theta)$ and $\sigma(\theta)$ satisfy appropriate conditions so that $\theta_t$ has a unique stationary distribution with a compact support.

Workers are endowed with one unit of labor. They are hand-to-mouth, in the sense they consume their current-period labor income. This assumption is made for simplicity and for the convenience of comparison to the existing literature which often makes the same simplifying assumption.

**Markets** There are perfectly competitive markets for labor and financial intermediation. A financial intermediary (or a bank) offers long-term lending contracts to entrepreneurs and has access to an inter-bank market in which all banks can borrow and lend at a risk-free interest rate $r$. Workers supply labor in the spot labor market at wage rate $w$.

**Contracts** A long-term lending contract $\mathscr{C}_{\tau}(\theta, K) \equiv \{C_{\tau+s}, I_{\tau+s}\}_{s=0}^{\infty}$ offered to an entrepreneur at date $\tau$ with productivity level $\theta_{\tau} = \theta$ specifies an initial capital $K_{\tau} = K$ and a sequence of future transfers to the entrepreneur, $\{C_{\tau+s}\}$ and the

[1]All our results extend to case when $M_t$ is a Levy process. In this section and the next, we assume $M_t$ is a standard Brownian motion to simplify notation. We will allow $M_t$ to contain jumps in Section 4.

process of cumulative investment $\{I_{\tau+s}\}$. Given $I_{\tau+s}$, the level of capital is determined by

$$dK_{\tau+s} = dI_{\tau+s} - \delta K_{\tau+s} ds,$$

where $dI_{\tau+s}$ is the amount of investment made at time $t$, and $\delta$ is the rate of depreciation.

Given a contract $\mathscr{C}_\tau(\theta, K)$, we can compute the value of the contract to the the entrepreneur and to the bank. Let $U_t(\mathscr{C}_\tau)$ be a monotonic transformation the utility that an entrepreneur obtains from contract $\mathscr{C}_\tau$ after period $t$

$$U_t(\mathscr{C}_\tau) = \left\{ E_t \left[ \int_0^\infty e^{-\beta s} \beta \frac{C_{t+s}^{1-\gamma}}{1-\gamma} ds \right] \right\}^{\frac{1}{1-\gamma}} \quad \forall t \geq \tau. \tag{3}$$

Given a interest rate $r$ and wage rate $w$, the banks' value from a contract $\mathscr{C}_\tau(\theta)$ after period $t$

$$V_t(\mathscr{C}_\tau) = E_t \left[ \int_0^\infty e^{-rs} dD_{t+s} \right] \quad \forall t \geq \tau,$$

where the cumulative cash flow $D_{t+s}$ equals flow profits from operating the technology net of transfers to the entrepreneur and net of earnings retained for reinvestment

$$dD_{t+s} = \max_{L_{t+s}} \left[ \left\{ (\theta_{t+s} K_{t+s})^\alpha L_{t+s}^{1-\alpha} - w L_{t+s} \right\} - C_{t+s} \right] ds - dI_{t+s}.$$

**Agency frictions** At any time $t \geq \tau$, an entrepreneur can default on the lending contract. A default entails two events: (i) the entrepreneur absconds with a fraction $\lambda^{-1} < 1$ of the capital stock under operation, and (ii) anonymously enters into contract with a new financial intermediary. Let $\bar{U}_t$ be the value of the outside option, that is, the value to entrepreneur at date $t$ if he defaults on its existing lending contract. To avoid default, the financial intermediary must ensure that any offered contract $\mathscr{C}_\tau$ satisfies

$$U_t(\mathscr{C}_\tau) \geq \bar{U}_t \quad \forall t \geq \tau \tag{4}$$

Inequality (4) is an incentive compatibility constraint which restricts the space of feasible contracts. Given a process for outside options $\left\{\bar{U}_t\right\}_{t\geq\tau}$, an *optimal contract* maximizes $V_\tau(\mathscr{C}_\tau)$ subject to (4).

In our setup outside options $\bar{U}_t$ are endogenous and pinned down by the competition in the lending market. In particular, an outcome of perfect competition is that entrepreneurs extract full surplus from a new lending relationship, and therefore, the outside value $\bar{U}_t$ is the maximum value that an entrepreneur with productivity $\theta_t$ can obtain on the lending market. It satisfies

$$\bar{U}_{\tau+s} = \max_{\tilde{\mathscr{C}}_\tau} \left\{ U_\tau : V_\tau\left(\tilde{\mathscr{C}}_\tau\right) \geq 0 \text{ s.t. } \tilde{\mathscr{C}}_\tau \text{ is optimal given } \left\{\bar{U}_t\right\}_{t\geq\tau} \right\} \quad \forall\tau, t \geq \tau \tag{5}$$

**Recursive formulation** As is standard in the dynamic contracting literature, we use promised utility of the entrepreneur $U_t$ as a state variable to characterize the optimal contract. Let $V(\theta, K, U)$ be the maximum value to the bank from contract to an entrepreneur with productivity $\theta$, initial level of capital $K$ and which delivers the entrepreneur a value of $U$. In Appendix A , we show that associated with $V(\theta, K, U)$, there exists a function $\bar{U}(\theta, K, U)$ such that

$$V\left(\theta, K, \bar{U}(\theta, K)\right) = 0 \tag{6}$$

such that (i) the value of outside option $\bar{U}_t = \bar{U}(\theta_t, K_t)$ and (ii) the value under the optimal contract $V_t = V(\theta_t, K_t, U_t)$ is a given by the following maximization problem.

$$V(\theta, K, U) = \max_{\{C_t, I_t, L_t, G_t, D_t\}} E\left[\int_0^\infty e^{-rt} dD_t\right] \tag{7}$$

subject to:

$$dD_t = \max_{L_t}\left[\left\{(\theta_t K_t)^\alpha L_t^{1-\alpha} - wL_t\right\} - C_t\right]dt - dI_t$$

$$U_t \geq \bar{U}\left(\theta_t, \frac{K_t}{\lambda}\right) \tag{8}$$

and laws of motion:

$$d\theta_t = \mu(\theta_t)dt + \sigma(\theta_t)dM_t; \quad \theta_0 = \theta. \tag{9}$$

$$dK_t = dI_t - \delta K_t dt;\ K_0 = K \tag{10}$$

$$dU_t = \left[\frac{\beta}{1-\gamma}\left(U_t - C_t^{1-\gamma}U_t^\gamma\right) + \frac{1}{2}\frac{G_t^2}{U_t}\right]dt + G_t dM_t \quad U_0 = U, \tag{11}$$

We label the maximization problem as *P1.* We use

$$P1_f(\theta, K, U) \equiv \{C(\theta, K, U), I(\theta, K, U), D(\theta, K, U), G(\theta, U, K)\}$$

to denote the policy functions of the maximization problem *P1.* The last equation (11) is a promise keeping constraint that ensures that the contract delivers $U$ and the policy function $G(\theta, U, K)$ specifies the sensitivity of continuation utility with respect to the process $dM_t$. The optimal contract can be constructed using policy functions to *P1.*

**Recursive competitive equilibrium with limited enforcement** A stationary competitive equilibrium with limited enforcement consists of i) an interest rate $r$ and a wage rate $w$; ii) value function $V(\theta, K, U)$ with associated policy functions $P1_f(\theta, K, U)$; iii) outside options $\bar{U}(\theta, K)$; and iv) a stationary distribution $\Phi(\theta, K, U)$, such that:

1. Given wage $w$, interest rate $r$ and outside options $\bar{U}(\theta, K)$, the value function $V(\theta, K, U)$ and policy functions $P1_f(\theta, K, U)$ solves *P1.*

2. The outside options $\bar{U}(\theta, K)$ satisfy (6).

3. Goods market, labor market, and the inter-bank lending market clear:

$$\begin{aligned}
\int [C(\theta, K, U) + \delta K + wL(\theta, K, U)]\, d\Phi(\theta, K, U) &= \int (\theta K)^{\alpha} L(\theta, K, U)^{1-\alpha}\, d\Phi(\theta, K, U); \\
\int L(\theta, K, U) &= 1; \\
\int D(\theta, K, U)\, d\Phi(\theta, K, U) &= 0
\end{aligned}$$

   In the first equation, $C(\theta, K, U)$ is the consumption of the entrepreneur, $\delta K$ is investment, which equals depreciation in steady state, and $wL(\theta, K, U)$ is the consumption of the hand-to-mouth workers. The third equation states that the net flows across all banks are balanced and equal zero in the aggregate.

4. The stationary distribution $\Phi(\theta, K, U)$ is consistent the law of motion of state variables implied by the optimal contract.

Exploiting the homotheticity of the production function and preferences, in Appendix A, we show that the there exists a solution to *P1* such that $V(\theta, K, U)$ satisfies a unique affine decomposition

**Lemma 1.** *There exists a process $u(\theta)$ such that $u(\theta) \geq 0$ and $u'(\theta) \geq 0$ and*

$$V(\theta, K, U) = K - u^{-1}(\theta)U \tag{12}$$

For the rest of the paper we focus on equilibria where $V(\theta, K, U)$ satisfies (12).

**A measure of capital allocation** Our main interest will be in studying allocation of capital across firms. For a given equilibrium with limited enforcement, we define the efficient level of output as

$$\begin{aligned} Y^* &= \max \int (\theta_i K_i)^\alpha L_i^{1-\alpha} di \\ \textit{subject to} &: \int K_i di \leq \int K d\Phi(\theta, K, U); \quad \int L_i di \leq 1. \end{aligned}$$

Note that this measure keeps the total level of capital same as the level of capital in the competitive equilibrium. The efficiency ratio is defined as

$$EF = \frac{\int (\theta K(\theta, K, U))^\alpha L(\theta, A)^{1-\alpha} d\Phi(\theta, K, U)}{Y^*}. \tag{13}$$

That is, $EF$ is our measure of the efficiency of capital (re) allocation. A value of 1 implies perfectly efficient allocation and a value below one is a measure of capital misallocation arising due to agency frictions. Our measure of misallocation is defined in the same way as in, for example, Hsieh and Klenow (2009) and Moll (2014).

# 3 Decentralization

In this section, we show that the equilibrium with optimal contracting described in the above section can be decentralized as a competitive equilibrium with Arrow securities and a collateral constraint that are commonly used in the applied literature.

In the decentralized economy, entrepreneurs borrow and lend using state-contingent deposits (Arrow securities) and rent capital in spot competitive markets. Let $A_t$ denote the financial assets, that is, the net claims of an entrepreneur at time $t$ with the financial intermediaries. The consumption and investment decisions of an entrepreneur with initial productivity $\theta_0 = \theta$ and assets $A_0 = A$ is given by

$$\max_{C_t, g_t, K_t} E_0 \left[ \int_0^\infty e^{-\beta t} \beta \frac{C_t^{1-\gamma}}{1-\gamma} dt \right]$$

$$\text{s.t. } dA_t = [rA_t + \Pi(\theta_t, A_t) - C_t] dt + \phi_t A_t dM_t \quad A_0 = A, \tag{14}$$

$$d\theta_t = \mu(\theta_t) dt + \sigma(\theta_t) dM_t; \quad \theta_0 = \theta. \tag{15}$$

where profits $\Pi$ are

$$\Pi(\theta, A) = \max_{K,L} \quad (1-\tau)\left(\theta^\alpha K^\alpha L^{1-\alpha} - wL - (r+\delta) K\right)$$

$$K \leq \lambda A \tag{16}$$

We label the above-stated problem as *P2*. Equation (14) is the budget constraint of the entrepreneurs in which $\Pi_t$ are the flow profits from operating the firm, and stochastic process $\phi_t$ denotes the returns from Arrow securities. Equation (16) is the collateral constraint that limits the use of capital in production as a scalar $\lambda \geq 1$ (common across all entrepreneurs) times the level of entrepreneurs' financial assets. For a given risk-free interest $r$ and wage rate $w$, problem *P2* can be formulated recursively using $(\theta, A)$ as the state variables. We use $P2_f(\theta, A) \equiv \{\mathtt{C}(\theta, A), \mathtt{L}(\theta, A), \mathtt{K}(\theta, A), \phi(\theta, A)\}$ to denote the policy functions of the entrepreneur's maximization problem.

Problem *P2* is almost identical to the one typically modeled in the literature of capital misallocation, for example, Moll (2014), with only one key difference: in *P2*, the process $\phi_t$ is unrestricted whereas in the model of Moll (2014) that assumes exogenously incomplete markets imposes $\phi_t = 0$.

We now define a stationary competitive equilibrium with Arrow securities and collateral constraints.

**Recursive competitive equilibrium with Arrow securities and collateral constraints** A stationary competitive equilibrium with Arrow securities and collateral constraint consists of i) an interest rate $r$ and a wage rate $w$; ii) policy functions $P2_f(\theta, A)$; and iii) the stationary distribution of types $\Psi(\theta, A)$, such that:

1. Given the equilibrium wage $w$ and the equilibrium interest rate $r$, the policy functions $P2_f(\theta, A)$ solve *P2*

2. Goods, labor and the rental markets for capital clear.

$$\begin{aligned}
\int \left[\mathtt{C}(\theta, A,) + \delta \mathtt{K}(\theta, A) + w\mathtt{L}(\theta, A)\right] d\Psi(\theta, A) &= \int \left(\theta \mathtt{K}(\theta, A)\right)^{\alpha} \mathtt{L}(\theta, A)^{1-\alpha} d\Psi(\theta, A); \\
\int \mathtt{L}(\theta, A) &= 1; \\
\int \left(\mathtt{K}(\theta, A) - A\right) d\Psi(\theta, A) &= 0
\end{aligned}$$

3. The stationary distribution $\Psi(\theta, A)$ is consistent the law of motion of wealth implied by the optimal policies.

The main result in this section is that any equilibrium with limited enforcement and long-term contracts can be implemented by an equilibrium with Arrow securities and collateral constraints. From Alvarez and Jermann (2000), we know that limited enforcement settings such as Kehoe and Levine (1993) have a decentralization with complete markets and borrowing constraints. However, in their setting the tightness of the borrowing constraints depends on the entire history of idiosyncratic shocks. On the the hand, we construct a decentralization in which the collateral constraint is linear in capital and its tightness $\lambda$ is common across all agents irrespective of the history of their shocks. Our decentralization relies on constructing a mapping between promised utility $U_t$ and financial wealth $A_t$ and then obtaining an equivalence between the incentive constraint (8) and the collateral constraint (16).

To get the mapping between $U_t$ and $A_t$,start from two present value relationships: (i) the value of the bank $V(K, \theta, U)$ in equation (7) which equals the initial capital $K$ plus the present discounted (at interest rate $r$) value of cashflows from the entrepreneur; and (ii) the budget constraint of the entrepreneur in equation (14), which implies that the present discounted (also at rate $r$) value of entrepreneur's consumption minus income from business equals $A_t$. Since both (i) and (ii) use present discounted values of profits less entrepreneur's consumption, we get

$$A_t = K_t - V\left(\theta_t, K_t, U_t\right).$$

Using the decomposition of $V\left(\theta, K, U\right)$ from Lemma (1), we see that

$$A_t = u^{-1}\left(\theta_t\right) U_t \tag{17}$$

This gives us the explicit mapping between $A_t$ and $U_t$. A noteworthy aspect of (17) is that the level of capital $K$ drops out and for a given $\theta$, it gives a bijection between $A$ and $U$.

Next, we obtain the equivalence between the incentive constraint and the collateral constraint. An immediate corollary of Lemma (1) is that $\bar{U}\left(\theta, K\right)$ equals

$$\bar{U}\left(\theta, K\right) = u(\theta)K$$

and therefore the incentive constraint (8) can be written as $U_t \geq u(\theta_t)\frac{K_t}{\lambda}$.

Now substitute for $U_t$ from equation (17) above to see that

$$U_t \geq u(\theta_t)\frac{K_t}{\lambda} \quad \text{iff} \quad \lambda A_t \geq K_t. \tag{18}$$

The reason why we are able to construct a tractable decentralization as compared to Alvarez and Jermann (2000) lies in the specification of the outside options. Alvarez and Jermann (2000) start from the Kehoe and Levine (1993) setup where outside options are exogenous and given by the value of autarky. In our setting value of the outside option is endogenous. In particular, the left-hand-side of (18) shows that both the value from being in the contract as well as the value of defaulting scale with the level of capital $K$ and $\theta$. Since all history dependence is ultimately encoded in $(K, \theta)$, it "cancels" from both sides and we obtain a simple collateral constraint on the right-hand-side of the (18). We summarize our decentralization in the following proposition.

**Theorem 2.** *Suppose the economy with limited enforcement has stationary equilibrium*

$$\mathcal{E} = \left\{ \{r, w\}, P1_f(\theta, K, U), \bar{U}(\theta, K), \Phi(\theta, K, U) \right\},$$

*where $\bar{U}(\theta, K) = \bar{u}(\theta) K$ is constant return to scale. There exists a one-to-one mapping $U = \mathcal{U}(\theta, A)$ such that*

$$\mathtt{E} = \{\{r, w\}, P2_f(\theta, A), \Psi(\theta, A)\}$$

*is a competitive equilibrium with Arrow securities and collateral constraints, where the policy functions $P2_f(\theta, A)$ are given by*

$$\begin{aligned}
\mathtt{K}(\theta, A) &= K + dI(\theta, K, U), && (19)\\
\mathtt{L}(\theta, A) &= L(\theta, K, U), && (20)\\
\mathtt{C}(\theta, A) &= C(\theta, K, U), && (21)\\
\phi(\theta, A) &= \frac{1}{V(\theta, K, U) - K}\left[V_U(\theta, K, U) G(\theta, K, U) + V_\theta(\theta, K, U)\sigma(\theta)\right], && (22)
\end{aligned}$$

# 4 Misallocation and persistence

In an economy with exogenously incomplete markets, Moll (2014) finds the efficiency of capital allocation to be increasing in the persistence of productivity shocks. In this section, show that with limited commitment, this is no longer true, and the efficiency of capital allocation is decreasing in persistence when entrepreneurs are risk averse.

To derivation closed form solutions, we assume that $\theta_t$ follows a two state Markov chain with state space $\{\theta_H, \theta_L\}$ and with an instantaneous switching rate of $\kappa$. Formally, the law of motion of $\theta$ can be described as

$$d\theta_t = (\theta_H - \theta_L)\left[-I_H(\theta_t) dN_{H,t} + I_L(\theta_t) dN_{L,t}\right]. \tag{23}$$

where $I_i(\theta)$ is the indicator function that takes value of 1 when $\theta = \theta_i$, for $i = H, L$. The processes $N_{H,t}$ and $N_{L,t}$ are independent Poisson processes with a common intensity $\kappa$. It is convenient to define $\rho(\kappa) = e^{-2\kappa}$ as the autocorrelation of the process.[2] In our setup, lowering the value of $\kappa$ allows us to increase the persistence of the productivity shock, $\rho(\kappa)$, while keeping the unconditional distribution of $\theta_t$ unchanged. Our main interest is to characterize the relationship between $EF$ and the persistence of the productivity shock $\theta_t$.

In light of Theorem 2, we use the decentralization with Arrow securities to derive the implications on misalloaction. In the case of two-state Markov chain, the law of motion of entrepreneur's financial wealth can be written as:

$$\begin{aligned} dA_t = [rA_t + \Pi_t - C_t]\,dt + I_H(\theta_t)\,A_t\phi_H\,[dN_{H,t} - \kappa dt] \\ +I_L(\theta_t)\,A_t\phi_L\,[dN_{L,t} - \kappa dt]\,. \end{aligned} \tag{24}$$

The profits $\Pi_t(\theta, A) = \lambda\pi(\theta)\,A$ after maximizing with respect to $K, L$. With this simplification, the budget constraint becomes linear in $A$ and so is the optimal consumption policy: $\frac{C(\theta,A)}{A} = c(\theta)$ for some $c(\theta)$. We characterize the entrepreneurs' consumption and saving policy below.

**Lemma 3.** *The value function of the entrepreneur's optimization problem is of the form* $V(\theta, A) = \frac{1}{1-\gamma}H(\theta)\,A^{1-\gamma}$, *where* $H(\theta_H) = \left\{\frac{1}{\gamma}\left[\beta + (\gamma-1)(r + \pi(\theta_H)) + \gamma\kappa(1-\omega)\right]\right\}^{-\gamma}$, *and* $H(\theta_L) = \left\{\frac{1}{\gamma}\left[\beta + (\gamma-1)(r + \lambda\pi(\theta_L)) + \gamma\kappa(1-\omega^{-1})\right]\right\}^{-\gamma}$. *The parameter* $\omega = \left[\frac{H(\theta_H)}{H(\theta_L)}\right]^{-\frac{1}{\gamma}}$ *and is given by*

$$\omega = \frac{\beta + (\gamma-1)\left[r + \lambda\pi(\theta_H)\right] + 2\gamma\kappa}{\beta + (\gamma-1)\left[r + \lambda\pi(\theta_L)\right] + 2\gamma\kappa}. \tag{25}$$

*The optimal consumption policy is* $c(\theta) = H(\theta)^{-\frac{1}{\gamma}}$, *and the optimal policy for* $\phi$ *is given by:*

$$\phi_H = \omega - 1, \quad \phi_L = \omega^{-1} - 1.$$

*Proof.* See Appendix B. □

Entrepreneurs are allowed to allocate their financial wealth across different states of the world, which is reflected in the policy function $\phi_H$ and $\phi_L$. By equation (24), entrepreneurs with high productivity pays $\phi_H\kappa A = (\omega-1)\,\kappa A$ per unit of time in

[2] It can be shown that the autocorrelation between $\theta_t$ and $\theta_{t+\tau}$ is $e^{-2\kappa\tau}$ under the stationary distribution.

exchange for an Arrow security that pays $\omega - 1$ fraction of wealth in the event of a low productivity shocks. From equation (25), $\omega - 1 > 0$ as long as $\gamma > 1$.

The payment $(\omega - 1)\,\kappa A$ can be interpreted as a premium charged for default risk. Because for high-productivity entrepreneurs, $K = \lambda A$ is binding, the flow payment for high productivity entrepreneurs, $(\omega - 1)\,\kappa A = (\omega - 1)\,\frac{\kappa}{\lambda} K$. Therefore, $(\omega - 1)\,\frac{\kappa}{\lambda}$ is the excess amount paid for borrowing capital $K_t$ relative to the standard user cost $r + \delta$. Thus $(\omega - 1)\,\frac{\kappa}{\lambda}$ can be interpreted as the compensation for default risk: in the event of a low productivity shock, the project is no-longer profitable ($\pi(\theta_L) = 0$) and the capital stock $K = \lambda A$ is liquidated, in which case the entrepreneur can retain a fraction $(\omega - 1)\,\frac{1}{\lambda}$ of the liquidation value of the asset.

Access to such financial markets in our model allows agents to trade off consumption risk sharing versus capital reallocation. The fact that $\omega > 1$ highlights the tradeoff between consumption efficiency and production efficiency. The need for risk sharing creates an incentive for agents to carry more financial wealth to the low productivity state, while production efficiency requires more wealth in the high productivity state in order to relax the borrowing constraint. Thus allowing for state contingent contracts allows agents to achieve a higher consumption efficiency, but at the cost of a lower production efficiency. In fact, as the productivity process becomes more persistent, the incentive for risk sharing is stronger, and measured capital misallocation in equilibrium is more severe.

To study the implications of our model on capital misallocation, we need to characterize the equilibrium distribution of financial wealth. Define $\mathbf{A}_{H,t} = \int A_{i,t} I_{\{\theta_{i,t} = \theta_H\}} di$ and $\mathbf{A}_{L,t} = \int A_{i,t} I_{\{\theta_{i,t} = \theta_L\}} di$ to be the total amount of wealth for high and low productivity entrepreneurs, respectively. As we show in Appendix B,

$$d\mathbf{A}_{H,t} \;=\; \left[r + \lambda\pi(\theta_H) - c(\theta_H) - \kappa\omega\right]\mathbf{A}_{H,t}dt + \kappa\omega^{-1}\mathbf{A}_{L,t}dt \tag{26}$$

The interpretation is that $r$ is interest payment, $\lambda\pi(\theta_H) - c(\theta_H)$ is the net income, which is the operating profit of the firm less the consumption of the entrepreneur, and $\kappa\omega$ is the premium paid in exchange for the insurance for the low-productivity shock. The term $\kappa\omega^{-1}\mathbf{A}_{L,t}dt$ reflect the fact that low productivity firms become high productivity ones at rate $\kappa$ and whenever they do so, they carry $\omega^{-1}$ fraction of their wealth into the high productivity state. Similarly,

$$d\mathbf{A}_{L,t} = \left[r + \lambda\pi(\theta_L) - x(\theta_L) - \kappa\omega^{-1}\right]\mathbf{A}_{L,t}dt + \kappa_H\omega\mathbf{A}_{H,t}dt \tag{27}$$

Define $\eta = \frac{\mathbf{A}_H}{\mathbf{A}_H + \mathbf{A}_L}$, where we use $\mathbf{A}_H$ and $\mathbf{A}_L$ for steady state levels of total entrepreneurial wealth in high and low productivity states, respectively. Stationarity requires $d\mathbf{A}_{H,t} = d\mathbf{A}_{L,t} = 0$. Equations (25), (26), and (27) are therefore three equations to be solved for three unknowns, $\omega$, $r$, and $\eta$.

Although there is no aggregate risk and we allow for state-contingent contracts in our economy, the presence of the financial constraint $K_t \leq \lambda A_t$ creates a tension between production efficiency and consumption risk sharing. The presence of incomplete risk sharing implies that the equilibrium interest rate $r$ is typically less than $\beta$. In fact, $r$ can be negative. Whenever this happens, equilibrium may fail to exist, because the life-time utility of the agent may not be finite. The following assumption imposes a lower bound on the discount rate so that interest rates are always non-negative and an equilibrium with misallocation exists.

**Assumption.** *The parameters of the model satisfy* $\beta > \frac{1}{2}\lambda\delta\left(\frac{\theta_H}{\theta_L} - 1\right)$ *and* $\lambda \in (1, 2)$.

Intuitively, if $\lambda > 2$ then the financial constraint is not tight enough and an equilibrium with perfect risk sharing and perfect capital allocation cannot be supported. Under our assumption, the stationary distribution puts $\frac{1}{2}$ probability on $\theta_H$ and $\frac{1}{2}$ probability on $\theta_L$. If $\lambda > 2$, then the high-productivity entrepreneurs alone will be able to absorb all capital stock of the economy. Because our main result is a comparative statics exercise with respect to risk aversion $\gamma$, the assumption that $\beta > \frac{1}{2}\lambda\delta\left(\frac{\theta_H}{\theta_L} - 1\right)$ is sufficient to guarantee that the life-time utility of the agents is finite for arbitrary values of $\gamma$. Our main theorem below summarizes the relationship between persistence and misallocation.

**Theorem 4.** *Under Assumption 4, for any* $\rho$, $\exists\gamma$ *such that* $\gamma > \bar{\gamma}$ *implies that there exists an equilibrium with misallocation and* $\frac{\partial(1-EF)}{\partial\rho} > 0$.

*Proof.* See Appendix C □

In Figure 1, we plot the implications of the two-state version of our model for different values of $\rho$. As $\rho$ increases, the productivity process becomes more persistent, and so does the income process of the entrepreneurs. To hedge the risk exposure, the agents in our economy need to allocate more financial assets to the low productivity state. However, this risk sharing incentive makes the financial constraint tighter for high-productivity entrepreneurs. As a result, as $\rho$ increases to $\rho^*$, $\omega$ rises, that is, the agent allocate a higher fraction of wealth to the low productivity state. As $\omega$ becomes larger, the fraction of wealth owned by high productivity entrepreneurs, $\eta$ reduces and capital misallocation worsens. In this case, $\eta$ is minimized at $\rho^*$.

As $\rho$ rises above $\rho^*$, $\omega$ keep increasing, indicating that financial constraint becomes tighter; however, $\eta$ starts to drop and capital misallocation improves. In our model, risk sharing exacerbates capital misallocation because it requires risk averse entrepreneurs to carry more financial wealth into the low productivity state. The total amount of financial wealth carried into the low productivity state is $\kappa\omega$. Therefore, as $\kappa$ decreases towards zero (that is, $\rho$ increases towards 1), the product $\kappa\omega$ first rises, which account for the drops in $\eta$ and rises in misallocation. Eventually, as $\rho$ increases above $\rho^*$, the impact of $\kappa$ dominates, and $\kappa\omega$ starts to decline. As a result, although the financial constraints become tighter, capital misallocation improves until $\rho$ reaches $\hat{\rho}$, where the wealth of the high productivity entrepreneurs is enough to accommodate all capital stock of the economy and capital misallocation drops to zero.

At $\hat{\rho}$, although capital misallocation drops to zero, the marginal product of capital for high productivity firms is still strictly higher than the use cost of capital, $r + \delta$, and interest rate is still strictly lower than discount rate $\beta$. As $\rho$ keeps increasing towards 1, the financial constraint relaxes and eventually, the economy converges to one without misallocation with perfect risk sharing, that is, $EF = 1$ and $\omega = 1$.

Theorem 4 states that for a fixed level of persistence, $\rho$, as risk aversion $\gamma$ increases, so does the need for risk sharing and eventually, capital misallocation increases with the persistence of productivity shock. Figure 1 indicates that for a fixed risk aversion $\gamma$, as the persistence of the productivity shock converges to 1, eventually, capital misallocation disappears and the model converges to a model with permanent productivity shocks. This is the mechanism emphasized in Moll (2014) which is a special case of our setting if we impose $\phi_H = \phi_L = 0$. This is not surprise, in a model with permanent productivity shocks, high productivity entrepreneurs will eventually save enough and grow out of the financial constraint. In our numerical examples, however, the risk sharing mechanism dominates for most levels of persistence, and this convergence happens only as the persistence of productivity shock $\rho$ becomes extremely close to 1.

# 5 Calibrated Economy

In the previous section, we used a simplified shock process where $\theta_t$ took only two values. This helped us with closed-form characterization of the wealth distribution and all equilibrium quantities. However, the insights are more general. In this section, we study them numerically using a more "standard" process for $\theta_t$. We demonstrate

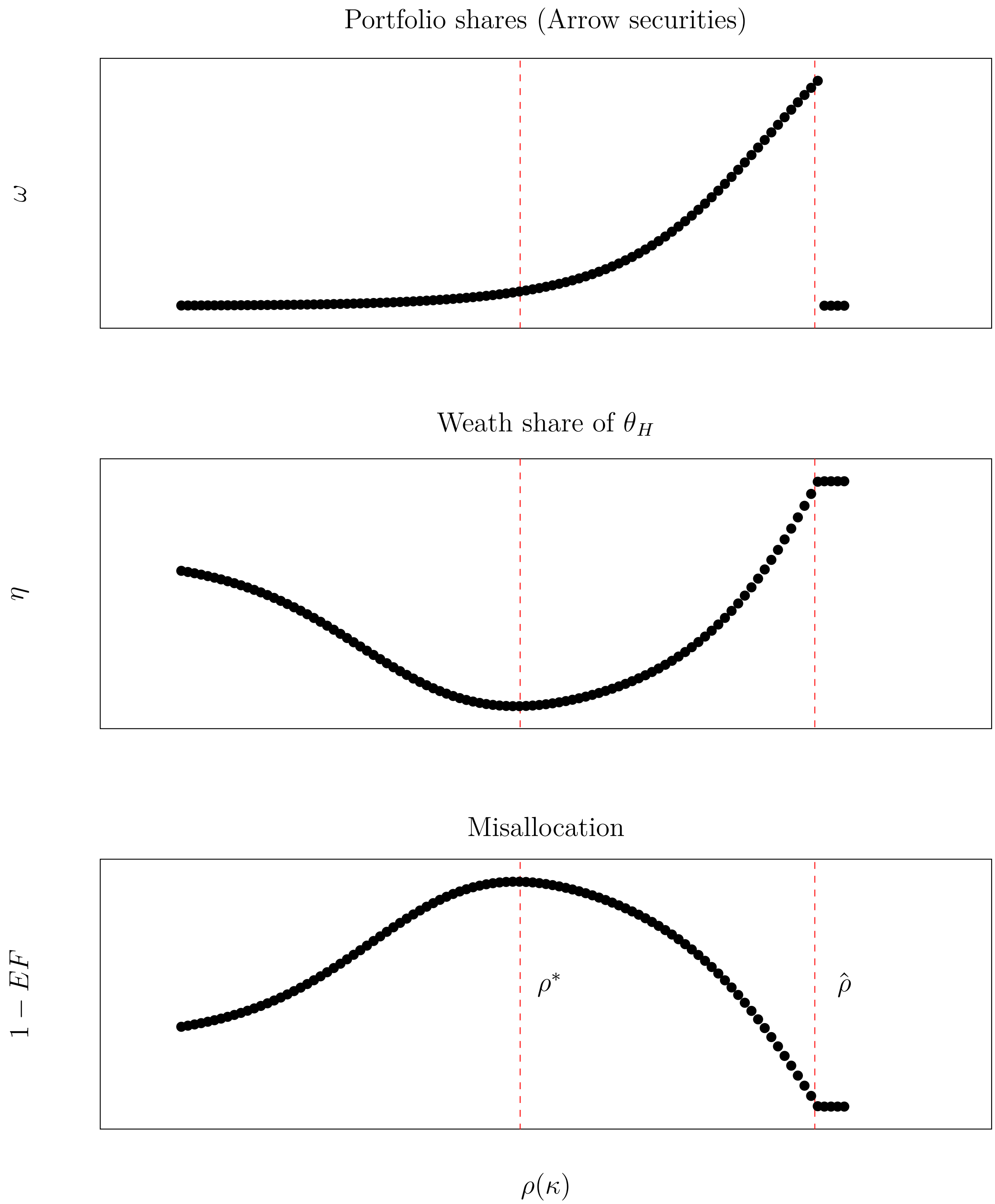


Figure 1: Allocations in the two-state economy with limited enforcement

| | | |
|---|---|---|
| Discount factor | $\beta$ | 0.05 |
| Capital share | $\alpha$ | 1/3 |
| Depreciation | $\delta$ | 0.05 |
| Collateral constrain | $\lambda$ | 1.2 |
| std. of idio risk | $\sigma$ | 0.76 |
| Persistence | $e^{-\kappa}$ | $(0,1)$ |
| Risk aversion | $\gamma$ | {3,5} |

Table 1: Calibration

that for reasonable levels of risk aversion, the patterns of misallocation with limited enforcement are quite different from those in the settings with exogenously incomplete markets used in the capital misallocation literature.

We let the log of the productivity $\theta$ follow

$$d\tilde{\theta}_t = \mu\left(\tilde{\theta}_t\right)dt + \sigma\left(\tilde{\theta}_t\right)dB_t,$$

where

$$\mu\left(\tilde{\theta}\right) = 2\kappa\left(\bar{\theta} - \tilde{\theta}\right); \quad \sigma\left(\tilde{\theta}\right) = \sqrt{\kappa}\sigma\left(\tilde{\theta}_H - \tilde{\theta}\right)\left(\tilde{\theta} - \tilde{\theta}_L\right).$$

The process has several desirable properties. The support of $\theta_t$ is bounded in $[\theta_L, \theta_H]$. The parameter $\kappa$ fully determines persistence and the stationary distribution of log productivity is independent of $\kappa$. We paramterize $\log\theta_L = -c\sigma$ and $\log\theta_H = c\sigma$ and calibrate the two parameters $\sigma$ and $c$ to target an unconditional standard deviation of $\log\theta$ to be 60% and the upper and lower bounds of the support of $\theta$ to be sufficiently large by setting $\frac{\theta_H - \theta_L}{\theta_L} = 8$. The rest of the parameters are standard and taken from Moll (2014). (See Table 1).

In Figure 2, we plot the resulting misallocation as a function $e^{-2\kappa}$ for several values of $\gamma$ for our economy with limited enforcement and then we compare outcomes to the economy with only a risk-free bond (or the market structure assumed in Moll (2014) ).

Qualitatively we see the same patterns as in Figure 1, confirming our theoretical insights. The bold and dashed lines depict the misallocation with limited enforcement and risk-free bond economy respectively. When the persistence parameter is low, both economies have a similar levels of misallocation of about 22%. The curves diverge when the persistence becomes high. The misallocation with risk-free bonds monotonically declines. The misallocation with limited enforcement is flatter for levels of persistence below 0.8 and then start increasing. When risk aversion is high, that is, $\gamma = 5$, it peaks to a level of more than 30% when the autocorrelation is about 0.99

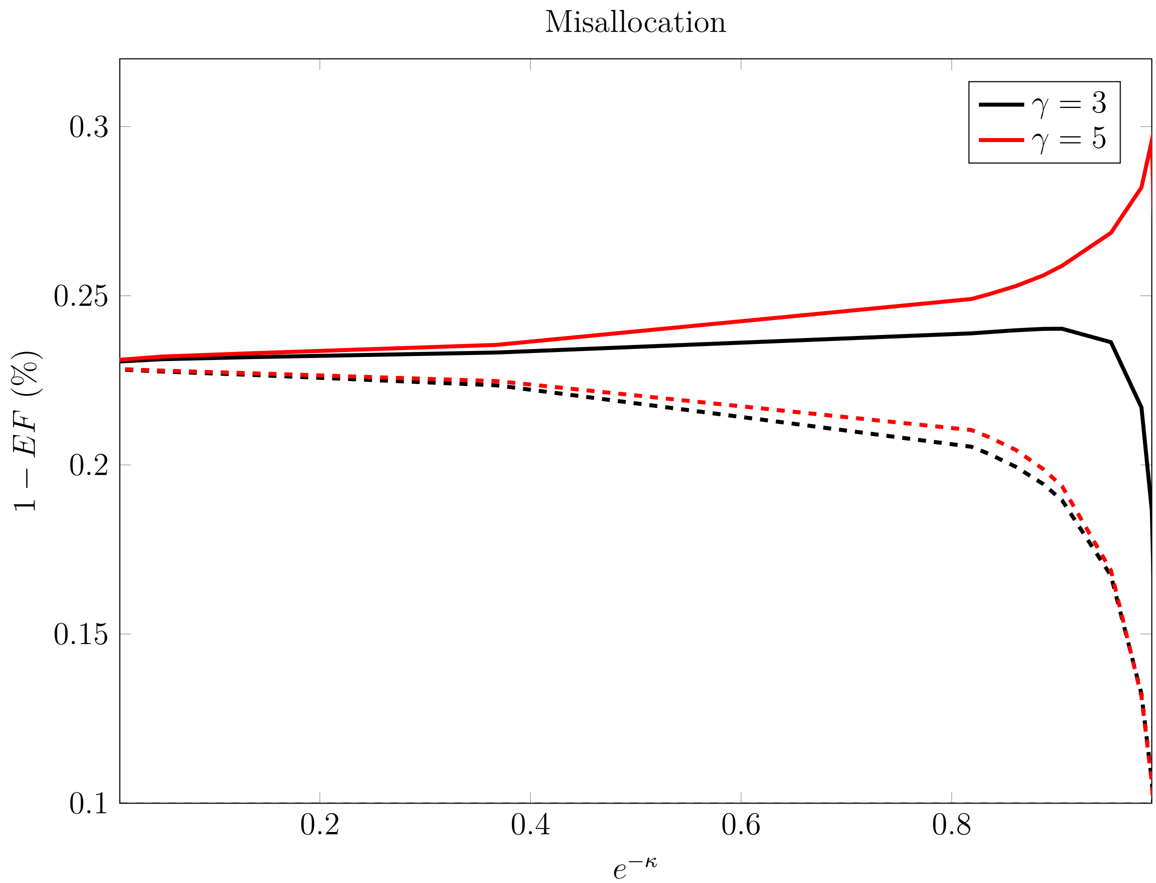


Figure 2: The figure plots the degree of misallocation. The solid (dashed) lines plots the degree of misallocation in the economy with limited enforcement (risk-free bond).

at which point it is about three times larger than the misallocation with a risk-free bond. [3]

# 6 Conclusion

This paper shows that factor misallocation is closely tied to the risk-sharing avenues available to firm owners. In contrast to the commonly studied bond-only economy with collateral constraints (for example Moll (2014)), we find that keeping fixed the nature of financial frictions, the degree of misallocation is *increasing* in persistence of the idiosyncratic risk when firms have access to state-contingent contracts. Allowing the possibility to transfer wealth from states where productivity is high to states where productivity is low generates a force that works against efficient allocation of capital. We show that for reasonable values of risk aversion, insurance needs more than offset efficiency concerns. A rigorous empirical examination of the extent of explicit and implicit insurance available to entrepreneurs will require us to study consumption patterns of firm-owners. We leave this for future work.

[3] The curves eventually converge to zero at $\kappa \to \infty$.

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# A Proof for Theorem 2

**The decentralized economy** First, note that in the competitive economy with collateral constraints, entrepreneur' optimization problem *P2* in the text is a standard convex programming problem, a neccessary and sufficient condition for optimality is that

$$\beta W(\theta, A) = max_{C,K,\phi}\Bigg\{\frac{1}{1-\gamma}\beta C^{1-\gamma} + [rA + \pi(\theta)K - C]W_A(\theta, A) \\ + \frac{1}{2}A^2\phi^2 W_{AA}(\theta, A) + \mu(\theta)W_\theta(\theta, A) + \frac{1}{2}\sigma^2(\theta)W_{\theta\theta}(\theta, A) + \sigma_\theta(\theta)\phi A W_{A,\theta}(\theta, A)\Bigg\}. \tag{28}$$

subject to

$$K \leq \lambda A \tag{29}$$

Define $U = [(1-\gamma)W]^{\frac{1}{1-\gamma}}$, then $W(\theta, A) = \frac{1}{1-\gamma}U^{1-\gamma}$. We have

$$\begin{aligned} W_A(\theta, A) &= U^{-\gamma}U_A(\theta, A); V_\theta(\theta, A) = U^{-\gamma}U_\theta(\theta, A); \\ W_{AA}(\theta, A) &= -\gamma U^{-\gamma-1}U_A(\theta, A)^2 + U^{-\gamma}U_{AA}(\theta, A); \\ W_{\theta\theta}(\theta, A) &= -\gamma U^{-\gamma-1}U_\theta(\theta, A)^2 + U^{-\gamma}U_{\theta\theta}(\theta, A); \\ W_{\theta A}(\theta, A) &= -\gamma U^{-\gamma-1}U_A(\theta, A)U_\theta(\theta, A) + U^{-\gamma}U_{\theta A}(\theta, A). \end{aligned}$$

The HJB in (28) is written as:

$$\begin{aligned} \beta\frac{1}{1-\gamma}\left[1 - \left(\frac{C}{U}\right)^{1-\gamma}\right] = &[rA + \pi(\theta)K - C]\frac{U_A}{U} + \mu_\theta(\theta)\frac{U_\theta}{U} \\ &+ \frac{1}{2}A^2\phi^2\left[-\gamma\left(\frac{U_A}{U}\right)^2 + \frac{U_{AA}}{U}\right] + \frac{1}{2}\sigma_\theta^2\left[-\gamma\left(\frac{U_\theta}{U}\right)^2 + \frac{U_{\theta\theta}}{U}\right] \\ &+ \phi\sigma_\theta A\left[-\gamma\frac{U_\theta}{U}\frac{U_A}{U} + \frac{U_{\theta A}}{U}\right]. \end{aligned}$$

Note that in our formulation, the utility maximization problem is homogenous of degree one. Therefore, $U(\theta, A) = u(\theta)A$ for some function $u(\theta)$. The above can be written as:

$$\beta \frac{1}{1-\gamma}\left[1-\left(\frac{C}{U}\right)^{1-\gamma}\right] = \left[rA+\pi\left(\theta\right)K-C\right]\frac{1}{A}+\mu_{\theta}\left(\theta\right)\frac{u'\left(\theta\right)}{u\left(\theta\right)} \\ -\frac{1}{2}\gamma\phi^{2}+\frac{1}{2}\sigma_{\theta}^{2}\left[-\gamma\left(\frac{u'\left(\theta\right)}{u\left(\theta\right)}\right)^{2}+\frac{u''\left(\theta\right)}{u\left(\theta\right)}\right]+\left(1-\gamma\right)\phi\sigma_{\theta}\frac{u'\left(\theta\right)}{u\left(\theta\right)} \tag{30}$$

**The contracting environment** In the economy with limited enforcement, the optimal contracting problme, which we will called *P1* in the text can be written as:

$$\begin{aligned} V\left(\theta,K,U\right) &= \max_{\{C_t,I_t,L_t,g_t\}} E\left[\int_0^{\infty} e^{-rt}\left\{\left[\left(\theta_t K_t\right)^{\alpha}L_t^{1-\alpha}-wL_t-C_t\right]dt-dI_t\right\}\right] \\ dK_t &= dI_t-\delta K_t dt;\ K_0=K \\ dU_t &= \left[\frac{\beta}{1-\gamma}\left(1-\left(\frac{C_t}{U_t}\right)^{1-\gamma}\right)+\frac{1}{2}g_t^2\right]U_t dt+g_tU_tdB_t,\ U_0=U \\ U_t &\geq \bar{U}\left(\theta_t,\frac{K_t}{\lambda}\right), \\ d\theta_t &= \mu(\theta_t)dt+\sigma(\theta_t)dM_t. \end{aligned} \tag{31}$$

Given the value function of the optimal contracting problem, the outside option satisfies

$$\bar{V}\left(\theta,K,\bar{U}\left(\theta,K\right)\right)=0. \tag{32}$$

First, maxing out $L_i$, the objective function can be written as $E\left[\int_0^{\infty} e^{-rt}\left\{\left[MPK\left(\theta_t\right)K_t-C_t\right]dt-d\right.\right.$ where $MPK\left(\theta\right)=\frac{1-\alpha}{w}\theta$. Next, we can rewrite the integral $\int_0^{\infty} e^{-rt}\left\{\left[MPK\left(\theta_t\right)K_t-C_t\right]dt-dI_t\right\}$ as

$$\int_0^{\infty} e^{-rt}\left\{\left[\left[MPK\left(\theta_t\right)-\left(r+\delta\right)\right]K_t-C_t\right]dt+\left(r+\delta\right)K_tdt-dI_t\right\}.$$

Using integration by parts, the last two terms can be writtten as

$$\begin{aligned} \int_0^{\infty} e^{-rt}\left\{\left(r+\delta\right)K_t-dI_t\right\} &= \int_0^{\infty} e^{-rt}rK_tdt+\int_0^{\infty} e^{-rt}\left\{\delta K_tdt-dI_t\right\} \\ &= \int_0^{\infty} e^{-rt}rK_tdt-\int_0^{\infty} e^{-rt}dK_t \\ &= -\left. e^{-rt}K_t\right|_0^{\infty} \\ &= K_0. \end{aligned}$$

This allows us to replace the control variable $dI_t$ by $K_t$ and write the optimal contracting problem as

$$
\begin{aligned}
V(\theta, K, U) &= \max_{\{C_t, K_t, g_t\}} E\left[\int_0^\infty e^{-rt}\left[\left[MPK(\theta_t) - (r+\delta)\right]K_t - C_t\right]dt + K\right] \\
dU_t &= \left[\frac{\beta}{1-\gamma}\left(1 - \left(\frac{C_t}{U_t}\right)^{1-\gamma}\right) + \frac{1}{2}g_t^2\right]U_t dt + g_t U_t dB_t,\ U_0 = U \qquad (33) \\
U_t &\geq \bar{U}\left(\theta_t, \frac{K_t}{\lambda}\right), \\
d\theta_t &= \mu(\theta_t)dt + \sigma(\theta_t)dM_t,\ \theta_0 = \theta.
\end{aligned}
$$

It is more convenient to work on the cost minimization problem than the profit maximization problem. We define $P(\theta, U)$ to be the value function of the following cost minimization problem, P2:

$$
\begin{aligned}
P(\theta, U) &= \min_{\{C_t, K_t, g_t\}} \int_0^\infty e^{-rt}\left[C_t - \pi(\theta_t)K_t\right]dt \\
dU_t &= \left[\frac{\beta}{1-\gamma}\left(1 - \left(\frac{C_t}{U_t}\right)^{1-\gamma}\right) + \frac{1}{2}\gamma g_t^2\right]U_t dt + g_t U_t dB_t,\ U_0 = U \\
U_t &\geq \bar{U}\left(\theta_t, \frac{K_t}{\lambda}\right), \\
d\theta_t &= \mu(\theta_t)dt + \sigma(\theta_t)dM_t,\ \theta_0 = \theta. \qquad (34)
\end{aligned}
$$

Clearly, $V(\theta, K, U) = K - P(\theta, U)$. The solution for the optimal contracting problem, P1 can therefore be constructed from (P2). We formally state our equivalence result as follows

**Lemma 5.** *(P1 and P2)*

*Given equilibrium price* $r, w$, $\{C(\theta, U, K), I(\theta, U, K), L(\theta, U, K), g(\theta, U, K)\}$ *solves the optimal contracting problem and* $\bar{U}(\theta, K)$ *satisfy (32) if and only if*

1. $\forall(\theta, U)$, $C(\theta, U) = C(\theta, U, K)$, $K(\theta, U) = K + dI(\theta, U, K)$, $g(\theta, U) = g(\theta, U, K)$ and $L(\theta, U) = \left(\frac{1-\alpha}{w}\right)^{\frac{1}{\alpha}}\theta K(\theta, U)$ are the policy functions for P2.
2. The outside option $\bar{U}(\theta, K)$ satisfies $K = P\left(\theta, \bar{U}(\theta, K)\right)$.

**Equivalence** Using the above lemma, we can construct a one-to-one mapping between the equilibrium in the economy with limited enforcement and the equilibriums in the decentralized economy. We let

$$\mathcal{A}(U|\theta) = P(\theta, U).$$

We need to prove that the policy functions in (34) constitute an equilibrium in the decentralized economy. It is enough to show that the following proposed policy functions satisfy the optimality conditions for enterpreneur's utility maximization problem.

$$\mathtt{K}(\theta, A) = K(\theta, U), \tag{35}$$

$$\mathtt{L}(\theta, A) = L(\theta, U), \tag{36}$$

$$\mathtt{C}(\theta, A) = C(\theta, U), \tag{37}$$

$$\phi(\theta, A) = \frac{1}{P(\theta, U)}\left[UP_U(\theta, U)\, g(\theta, U) + P_\theta(\theta, U)\,\sigma(\theta)\right], \tag{38}$$

where $U = \mathcal{U}(A|\,\theta)$ in the above formulas. Note that given the equivalence between P1 and P2 in Lemma 5, the right hand side of the policy functions in (35)-(38) are the same as those in (19)-(22).

Because the utility maximization problem in the decentralized economy is a well-defined convex programming problem, we only need to show that the proposed policy function satisfies the optimality conditions of the utility maximization problem, (28). The optimality condition for the contracting problem is:

$$\begin{aligned} rP(\theta, U) = min_{C,g,K}\Bigg\{ & C - \pi(\theta, K) + \left[\frac{\beta}{1-\gamma}\left[1 - \left(\frac{C}{U}\right)^{1-\gamma}\right] + \frac{1}{2}\gamma g^2\right] UP_U(\theta, U) \\ & + \underbrace{\frac{1}{2}g^2U^2P_{UU}(\theta, U)\,\frac{1}{2}g^2U^2}_{\textbf{[is this correct]}} + \frac{1}{2}\sigma_\theta^2 P_{\theta\theta}(\theta, U) + Ug\sigma_\theta P_{\theta U}(\theta, U)\Bigg\}. \\ s.t.\ U \geq \bar{U}\left(\theta, \frac{K}{\lambda}\right), \end{aligned} \tag{39}$$

Let $C(\theta, U)$, $g(\theta, U)$ and $K(\theta, U)$ be the policy functions of the optimal contracting problem. We will show that the policy functions in (35) satisfy the HJB equation (28) with $A = P(\theta, U)$.

We focus on equilibrium in which both the value function $P(\theta, U)$ and the outside option $\bar{U}(\theta, K)$ are homogenous. Under our assumption that $\bar{U}(\theta, K) = \bar{u}(\theta)K$ is constant return to scale, so must be the cost function, that is, $P(\theta, U) = p(\theta)U$ for some $p(\theta)$. We first show that the constraint on $K$ in (28) and (39) are identical. Using the result of Lemma 5, $K = P\left(\theta, \bar{U}(\theta, K)\right)$ can be written as $K = p(\theta)\,\bar{u}(\theta)\,K$, therefore, $\bar{u}(\theta)$ and $p(\theta)$ must be related by $\bar{u}(\theta) = \frac{1}{p(\theta)}$ and the constraint $U \geq \bar{U}\left(\theta, \frac{K}{\lambda}\right)$ can be written as $K \leq \lambda p(\theta)\,U$. Note that the CRS property implies that the mapping $A = P(\theta, U)$ can be written as $A = P(\theta)\,U$, which establishes the equivalence between the limited enforcement constraint (39) and the collateral constraint (29).

It remains to show that the HJB in (39) implies that HJB in (30) for an investor with $U = U(\theta, A)$. Using the homogeneity property, it is straightforward to show that

$$\begin{aligned}
&P_U(\theta, U) = p(\theta); \quad P_{UU}(\theta, U) = 0; \\
&P_\theta(\theta, U) = p'(\theta) U; \quad P_{\theta\theta}(\theta, U) = p''(\theta) U \\
&P_{\theta U}(\theta, U) = p'(\theta),
\end{aligned}$$

Using the above, dividing both sides by $UP_U(\theta, U) = p(\theta) U$, we can write the HJB (39) as

$$\begin{aligned}
0 = min_{C,g,K} \Bigg\{ &[C - \pi(\theta, K) - rp(\theta) U] \frac{1}{p(\theta) U} + \frac{\beta}{1-\gamma}\left[1 - \left(\frac{C}{U}\right)^{1-\gamma}\right]. \\
&+ \mu(\theta) \frac{p'(\theta)}{p(\theta)} + \frac{1}{2}\gamma g^2 + \frac{1}{2}\sigma^2(\theta) \frac{p''(\theta)}{p(\theta)} + g\sigma(\theta) \frac{p'(\theta)}{p(\theta)} \Bigg\}
\end{aligned}$$

Because $U = U(\theta, A)$, our mapping $A = P(\theta, U)$ implies that $A = p(\theta) u(\theta) A$. That is, $p(\theta) = \frac{1}{u(\theta)}$. In addition, using equation (38), we have $g = \frac{u'(\theta)}{u(\theta)}\sigma(\theta) + \phi$. Using this relationship to replace $p(\theta)$in the above equation, we have

$$\begin{aligned}
0 = min_{C,\phi,K} \Bigg\{ &[C - \pi(\theta) K - rA] \frac{1}{A} + \frac{\beta}{1-\gamma}\left[1 - \left(\frac{C}{U}\right)^{1-\gamma}\right] - \mu(\theta) \frac{u'(\theta)}{u(\theta)}. \\
&+ \frac{1}{2}\gamma \left[\left(\frac{u'(\theta)}{u(\theta)}\right)^2 \sigma^2(\theta) + \phi^2 + 2\phi\sigma_\theta \frac{u'(\theta)}{u(\theta)}\right] \\
&+ \frac{1}{2}\sigma_\theta^2 \left[2\left(\frac{u'(\theta)}{u(\theta)}\right)^2 - \frac{u''(\theta)}{u(\theta)}\right] - \frac{u'(\theta)}{u(\theta)}\sigma(\theta)\left[\frac{u'(\theta)}{u(\theta)}\sigma(\theta) + \phi\right] \Bigg\},
\end{aligned}$$

which is equivalent to (30) after simplification.

# B Proof for Lemma 3

**Entrepreneur optimization** Given the linearity of the profit function, the value function will be of the form: $V(\theta, A) = \frac{1}{1-\gamma} H(\theta) A^{1-\gamma}$. The HJB equation can be written as

$$\beta\frac{1}{1-\gamma}H(\theta_H)A^{1-\gamma} = \frac{1}{1-\gamma}C^{1-\gamma}+H(\theta_H)A^{1-\gamma}\left[r+\lambda\pi(\theta_H)-\frac{C}{A}-\kappa_H\phi_H\right] +\kappa_H\left[\frac{1}{1-\gamma}H(\theta_L)((1+\phi_H)A)^{1-\gamma}-\frac{1}{1-\gamma}H(\theta_H)A^{1-\gamma}\right];$$

and

$$\beta\frac{1}{1-\gamma}H(\theta_L)A^{1-\gamma} = \frac{1}{1-\gamma}C^{1-\gamma}+H(\theta_L)A^{1-\gamma}\left[r+\lambda\pi(\theta_L)-\frac{C}{A}-\kappa_L\phi_L\right] +\kappa_L\left[\frac{1}{1-\gamma}H(\theta_H)((1+\phi_L)A)^{1-\gamma}-\frac{1}{1-\gamma}H(\theta_L)A^{1-\gamma}\right];$$

As in the paper, we denote normalized consumption as $c=\frac{C}{A}$. The first order conditions imply

$$c(\theta)=H(\theta)^{-\frac{1}{\gamma}};\quad 1+\phi_H=\left[\frac{H(\theta_H)}{H(\theta_L)}\right]^{-\frac{1}{\gamma}};\;\; 1+\phi_L=\left[\frac{H(\theta_L)}{H(\theta_H)}\right]^{-\frac{1}{\gamma}}. \tag{40}$$

We can combine the HJB equations for $\theta_H$ and $\theta_L$ to get

$$\left[\frac{H(\theta_H)}{H(\theta_L)}\right]^{-\frac{1}{\gamma}}=\frac{\beta+(\gamma-1)(r+\lambda\pi(\theta_H))+\gamma\kappa_H\left[1-\left(\frac{H(\theta_H)}{H(\theta_L)}\right)^{-\frac{1}{\gamma}}\right]}{\beta+(\gamma-1)(r+\lambda\pi(\theta_L))+\gamma\kappa_L\left[1-\left(\frac{H(\theta_L)}{H(\theta_H)}\right)^{-\frac{1}{\gamma}}\right]}.$$

Define $\omega=\left[\frac{H(\theta_H)}{H(\theta_L)}\right]^{-\frac{1}{\gamma}}$, the above equation can be rearranged to get the expression for $\omega$ in (25). Clearly, given $\omega$, we can use Equation (40) to construct the policy functions stated in Lemma 3.

**Market clearing and equilibrium** We first derive an expression for the dynamics of $\mathbf{A}_{H,t}$ and $\mathbf{A}_{L,t}$. Under the optimal policy in Lemma 3, the law motion of the wealth of a high productivity entrepreneur is:

$$\frac{dA_t}{A_t}=[r+\pi(\theta_H)-c(\theta_H)+(\omega-1)\kappa]\,dt+(\omega-1)\,dN_{H,t},$$

and that of a low productivity entrepreneur is :

$$\frac{dA_t}{A_t}=\left[r+\pi(\theta_L)-c(\theta_L)+\left(\omega^{-1}-1\right)\kappa\right]dt+\left(\omega^{-1}-1\right)dN_{L,t}.$$

Given the above wealth dynamics, during a small interval $(t, t+\Delta)$, a $e^{-\kappa\Delta}$ fraction of entrepreneurs will remain at $\theta_{t+\Delta} = \theta_H$, the total wealth of all enterpreneurs who remains in $\theta_H$ becomes

$$\mathbf{A}_{H,t} e^{[r+\lambda\pi(\theta_H)-c(\theta_H)-\kappa(\omega-1)]\Delta}.$$

At the same time, $\left(1-e^{-\kappa\Delta}\right)$ of $\mathbf{A}_{L,t}$ experiences a regime switch and become $\theta_H$. When they beome $\theta_H$, $A_t$ changes to $\omega^{-1}A_t$. We have:

$$\begin{aligned} d\mathbf{A}_{H,t} &= \left[r+\lambda\pi\left(\theta_H\right)-c\left(\theta_H\right)-\kappa\left(\omega-1\right)-\kappa\right]\mathbf{A}_{H,t}dt+\kappa\omega^{-1}\mathbf{A}_{L,t}dt \\ &= \left[r+\lambda\pi\left(\theta_H\right)-c\left(\theta_H\right)-\kappa\omega\right]\mathbf{A}_{H,t}dt+\kappa\omega^{-1}\mathbf{A}_{L,t}dt. \end{aligned} \tag{41}$$

Similarly,

$$d\mathbf{A}_{L,t} = \left[r+\lambda\pi\left(\theta_L\right)-c\left(\theta_L\right)-\kappa\omega^{-1}\right]\mathbf{A}_{L,t}dt+\kappa\omega\mathbf{A}_{H,t}dt \tag{42}$$

The system can then be written as:

$$\begin{aligned} \frac{d\mathbf{A}_{H,t}}{\mathbf{A}_{H,t}} &= \left\{\left[r+\lambda\pi\left(\theta_H\right)-c\left(\theta_H\right)-\kappa\omega\right]+\kappa\omega^{-1}\frac{\mathbf{A}_{L,t}}{\mathbf{A}_{H,t}}\right\}dt, \\ \frac{d\mathbf{A}_{L,t}}{\mathbf{A}_{L,t}} &= \left\{\left[r+\lambda\pi\left(\theta_L\right)-c\left(\theta_L\right)-\kappa\omega^{-1}\right]+\kappa\omega\frac{\mathbf{A}_{H,t}}{\mathbf{A}_{L,t}}\right\}dt. \end{aligned}$$

We define $\eta = \frac{\mathbf{A}_H}{\mathbf{A}_H+\mathbf{A}_L}$ to be the faction of asset in the hands of high-productivity entrepreneurs. Imposing steady state, the above implies:

$$\left[r+\lambda\pi\left(\theta_H\right)-c\left(\theta_H\right)-\kappa\omega\right]+\kappa\omega^{-1}\frac{1-\eta}{\eta} = \left[r+\lambda\pi\left(\theta_L\right)-c\left(\theta_L\right)-\kappa\omega^{-1}\right]+\kappa\omega\frac{\eta}{1-\eta} = 0. \tag{43}$$

**Characterizing the equilibrium** Using the optimal consumption policy in Lemma 3,

$$c\left(\theta_H\right) = \frac{1}{\gamma}\left[\beta-r-\lambda\pi\left(\theta_H\right)\right]+\left[r+\lambda\pi\left(\theta_H\right)\right]+\kappa\left(1-\omega\right); \tag{44}$$

$$c\left(\theta_L\right) = \frac{1}{\gamma}\left[\beta-r-\lambda\pi\left(\theta_L\right)\right]+\left[r+\lambda\pi\left(\theta_L\right)\right]+\kappa\left(1-\omega^{-1}\right). \tag{45}$$

Using the above Equations, we can rewrite the (43) requirement as

$$\kappa\omega^{-1}\frac{1-\eta}{\eta} = \frac{1}{\gamma}\left[\beta - r - \lambda\pi\left(\theta_H\right)\right] + \kappa, \tag{46}$$

$$\kappa\omega\frac{\eta}{1-\eta} = \frac{1}{\gamma}\left[\beta - r - \lambda\pi\left(\theta_L\right)\right] + \kappa. \tag{47}$$

Combining Equations (46) and (47), we have

$$\left\{\frac{1}{\gamma}\left[\beta - r - \lambda\pi\left(\theta_H\right)\right] + \kappa\right\}\left\{\frac{1}{\gamma}\left[\beta - r - \lambda\pi\left(\theta_L\right)\right] + \kappa\right\} = \kappa^2.$$

In any equilibrium with misallocation, $\pi\left(\theta_L\right) = 0$. The above can therefore be simplified to a quadratic equation that determines $r$:

$$\left[\beta - r - \lambda\pi\left(\theta_H\right)\right]\left(\beta - r\right) + \gamma\left[2\left(\beta - r\right) - \lambda\pi\left(\theta_H\right)\right]\kappa = 0. \tag{48}$$

To summarize, Equation (48) determines equilibrium interest rate $r$. Given $r$, Equation (25) can be used to determine $\omega$. Given $r$ and $\omega$, we can then use equation (47) to solve for $\eta$ and fully determine the equilibrium.

$$\frac{\eta}{1-\eta} = \frac{\beta - r + \gamma\kappa}{\gamma\kappa\omega}. \tag{49}$$

To construct the rest of the equilibriu quantities, we first solve for the total capital stock of the economy. Note that because $MPK\left(\theta_L\right) = \alpha\theta_L\left(\int\theta_i K_i di\right)^{\alpha-1} = r + \delta$, we have $\int\theta_i K_i di = \left(\frac{\alpha\theta_L}{r+\delta}\right)^{\frac{1}{1-\alpha}}$. Let $\boldsymbol{K}$be the total capital stock, we have $\int\theta_i K_i di = \left(\eta\lambda\theta_H + \left(1-\eta\lambda\right)\theta_L\right)\boldsymbol{K}$. Therefore,

$$\boldsymbol{K} = \left(\frac{\alpha}{r+\delta}\right)^{\frac{1}{1-\alpha}}\frac{\theta_L^{\frac{1}{1-\alpha}}}{\eta\lambda\theta_H + \left(1-\eta\lambda\right)\theta_L}.$$

Finally, to compute the efficiency measure, we use Equation (13). Optimality on the labor market implies $L_i$ must be proportional to $\theta_i K_i$. Labor market clearing then implies $L_i = \frac{\theta_i K_i}{\int\theta_i K_i di}$. Therefore, $\int\left(\theta_i K_i\right)^\alpha L_i^{1-\alpha} di = \int\left(\theta_i K_i\right)^\alpha\left(\frac{\theta_i K_i}{\int\theta_i K_i di}\right)^{1-\alpha} di = \left[\int\theta_i K_i di\right]^\alpha$, and

$$EF = \frac{\left[\int\theta_i K_i di\right]^\alpha}{\theta_H^\alpha\boldsymbol{K}^\alpha} = \left[\frac{\theta_L\left(1-\lambda\eta\right) + \theta_H\lambda\eta}{\theta_H}\right]^\alpha = \left[\lambda\eta + \frac{\theta_L}{\theta_H}\left(1-\lambda\eta\right)\right]^\alpha. \tag{50}$$

# C Proof for Theorem 4

To study the comparative statics of the misallocation with respect to $\kappa$, note that if an equilibrium with misallocation exists, the solution can be constructed from Equations (48), (25) and (49). Equation (48) is a quadratic equation in $r$. The following lemma shows only the smaller root of the quadratic equation can be an equilibrium.

**Lemma 6.** *In any stationary equilibrium with misallocation, $\beta > r$.*

*Proof.* Equations (46) and (47) imply

$$\frac{\eta}{1-\eta} = \frac{\beta - r + \gamma\kappa}{\gamma\kappa\omega} = \frac{\gamma\kappa\omega^{-1}}{\beta - r + \gamma\kappa - \lambda\pi\left(\theta_H\right)}. \tag{51}$$

In any equilibrium with misallocation, $\pi\left(\theta_H\right) > 0$, the only way the above equation can simultaneously hold is $\beta > r$. If $\beta < r$ the first part implies $\frac{\eta}{1-\eta} < \frac{1}{\omega}$ and the second part implies $\frac{\eta}{1-\eta} > \frac{1}{\omega}$. $\square$

To prove Theorem 4, we first show that under Assumption 4, an equilibrium with misallocation exists for large enough $\gamma$. We start by establishing that the equilibrium interest rate must be positive and agents' life-time utility must be finite. Using $\pi\left(\theta_H\right) = \left(\frac{\theta_H}{\theta_L} - 1\right)(r+\delta)$ Equation (48) can be written as $\Phi\left(r\right) = 0$, where

$$\begin{aligned}\Phi\left(r\right) = &\left(1 + \lambda\left(\frac{\theta_H}{\theta_L} - 1\right)\right) r^2 - \left[2\left(\beta + \gamma\kappa\right) + \lambda\left(\frac{\theta_H}{\theta_L} - 1\right)\left(\beta + \gamma\kappa - \delta\right)\right] r \\ &+ \beta^2 + 2\beta\gamma\kappa - \lambda\delta\left(\frac{\theta_H}{\theta_L} - 1\right)\left(\beta + \gamma\kappa\right).\end{aligned}$$

Because $\Phi\left(\beta\right) = -\lambda\phi\gamma\kappa\left(\beta + \delta\right) < 0$, the larger of the two roots of $\Phi\left(r\right) = 0$ must satisfy $r > \beta$. By Lemma 6, the equilibrium interest rate must be the smaller of the two roots.

Under Assumption 4, for large enough risk aversion, the smaller root of $\Phi\left(r\right) = 0$ satisfies $0 < r < \beta$. This is straightforward because $\Phi\left(0\right) = \beta\left(\beta - \lambda\delta\left(\frac{\theta_H}{\theta_L} - 1\right)\right) + \gamma\kappa\left(2\beta - \lambda\delta\left(\frac{\theta_H}{\theta_L} - 1\right)\right) > 0$ for a fixed $\kappa$ and for large enough $\gamma$. Under the condition $0 < r < \beta$, the life-time utility of the entrepreneur is finite, $\omega > 1$ and $H\left(\theta\right) > 0$, where $H\left(\theta\right)$ is defined in Lemma 3.

Next, we show that for a fixed $\kappa$, for large enough risk aversion, the equilibrium always features capital misallocation, therefore, if we select the smaller root of $\Phi\left(r\right) = 0$, Equations (25) and (49) always define a valid equilibrium. To establish misallocation, it is enough to verify that the solution to (49) satisfies $\eta\lambda < 1$ so that not all capital are deployed by high-productivity entrepreneurs. Using Equation (49), it is enough to prove

$$\frac{\beta - r + \gamma\kappa}{\gamma\kappa\omega} < \frac{1}{\lambda - 1}, \tag{52}$$

as $\eta\lambda < 1$ is equivalent to $\frac{\eta}{1-\eta} < \frac{1}{\lambda-1}$. Fixing $\kappa$, for large enough $\gamma$, Equation implies 25 implies that $\omega \to \frac{r+2\kappa+\lambda\left(\frac{\theta_H}{\theta_L}-1\right)(r+\delta)}{r+2\kappa}$. Therefore, for $\gamma$ large enough, $\omega > 1+\epsilon$, where $\epsilon = \frac{\lambda\left(\frac{\theta_H}{\theta_L}-1\right)\delta}{\beta+2\kappa} > 0$, which implies that $\gamma$ large, $\frac{\beta-r+\gamma\kappa}{\gamma\kappa\omega} < 1 < \frac{1}{\lambda-1}$.

Having established that solutions to Equations (48), (25) and (49) are sufficient for a misallocation equilibrium for large enough $\gamma$, we now derive the comparative statics of $EF$ with respect to $\kappa$. We think of Equation (48) as defining $r$ as a function of $\kappa$. Given $r(\kappa)$, Equation (25) as defining $\omega$ as a function of $\kappa$, $\omega(\kappa)$, and Equation (49) as defining $\eta$ as a function of $\kappa$. The statement of the theorem is equivalent to $\frac{\partial ln\hat{\eta}}{\partial\kappa} > 0$, where $\hat{\eta} = \frac{\eta}{1-\eta}$. By Equation (49), $\hat{\eta}(\kappa) = \frac{\beta-r(\kappa)+\gamma\kappa}{\gamma\kappa\omega(\kappa)}$, and

$$\frac{d\ln\hat{\eta}}{d\kappa} = \frac{\gamma - r'(\kappa)}{\beta - r + \gamma\kappa} - \frac{1}{\kappa} - \frac{\omega'(\kappa)}{\omega}. \tag{53}$$

Note that $r'(\kappa)$ is bounded for any $\kappa$ and $\gamma$. For a fixed $\kappa$, let $\gamma \to \infty$, the first two terms are bounded, but $\omega'(\kappa) \to -\infty$ by Equation (25).